CLOSED ORBITS OF A CHARGE IN A WEAKLY EXACT MAGNETIC FIELD

WILL J. MERRY

ABSTRACT. We prove that for a weakly exact magnetic system on a closed connected Riemannian manifold, almost all energy levels contain a closed orbit. More precisely, we prove the following stronger statements. Let (M,g) denote a closed connected Riemannian manifold and $\sigma \in \Omega^2(M)$ a weakly exact 2-form. Let $\phi_t : TM \to TM$ denote the magnetic flow determined by σ, and let $c(g,\sigma) \in \mathbb{R} \cup \{\infty\}$ denote the Mañé critical value of the pair (g,σ). We prove that if $k > c(g,\sigma)$, then for every non-trivial free homotopy class of loops on M there exists a closed orbit of ϕ_t with energy k whose projection to M belongs to that free homotopy class. We also prove that for almost all $k < c(g,\sigma)$ there exists a closed orbit of ϕ_t with energy k whose projection to M is contractible. In particular, when $c(g,\sigma) = \infty$ this implies that almost every energy level has a contractible closed orbit. As a corollary we deduce that a weakly exact magnetic flow with $[\sigma] \neq 0$ on a manifold with amenable fundamental group (which implies $c(g,\sigma) = \infty$) has contractible closed orbits on almost every energy level.

1. INTRODUCTION

Let (M,g) denote a closed connected Riemannian manifold, with tangent bundle $\pi : TM \to M$ and universal cover $\tilde{M}$. We will assume M is *not* simply connected, as otherwise $\tilde{M} = M$ and all results proved in this paper reduce to special cases of the results in [5]. Let $\sigma \in \Omega^2(M)$ denote a closed 2-form, and let $\tilde{\sigma} \in \Omega^2(\tilde{M})$ denote its pullback to the universal cover. In this paper we consider the case where σ is *weakly exact*, that is, when $\tilde{\sigma}$ is exact (this is equivalent to requiring that $\sigma|_{\pi_2(M)} = 0$), however we do *not* assume that $\tilde{\sigma}$ necessarily admits a *bounded* primitive.

Let ω_g denote the standard symplectic form on TM obtained by pulling back the canonical symplectic form $dq \wedge dp$ on T^*M via the Riemannian metric. Let $\omega := \omega_g + \pi^*\sigma$ denote the *twisted symplectic form* determined by the pair (g,σ). Let $E : TM \to \mathbb{R}$ denote the energy Hamiltonian $E(q,v) = \frac{1}{2}|v|^2$. Let $\phi_t : TM \to TM$ denote the flow of the symplectic gradient of E with respect to ω; ϕ_t is known as a *twisted geodesic flow* or a *magnetic flow*. The reason for the latter terminology is that this flow can be thought of as modeling the motion of a particle of unit mass and unit charge under the effect of a magnetic field represented by the 2-form σ. Given $k \in \mathbb{R}^+ := \{t \in \mathbb{R} : t > 0\}$, let $\Sigma_k := E^{-1}(k) \subseteq TM$.

There exists a number $c = c(g,\sigma) \in \mathbb{R} \cup \{\infty\}$ called the *Mañé critical value* (see [14, 6, 7, 3] or Section 2 below for the precise definition) such that the dynamics of the hypersurface Σ_k differs dramatically depending on whether $k < c$, $k = c$ or $k > c$. Moreover $c < \infty$ if and only if

$\tilde{\sigma}$ admits a bounded primitive.

In this paper we study the old problem of the existence of closed orbits on prescribed energy levels. In the case when σ is exact, this problem has been essentially solved by Contreras in [5]; see Theorem D in particular, which gives contractible closed orbits in almost every energy level below the Mañé critical value, and closed orbits in every free homotopy class for any energy level above the critical value. In the case of surfaces a stronger result is known to hold: Contreras, Macarini and Paternain have proved in [9, Theorem 1.1] that in this case *every* energy level admits a closed orbit. However the case of a *magnetic monopole* (i.e. when σ is not exact) remains open, although much progress has been made. Let us describe now some of these results. A more comprehensive survey can be found in the introduction to [9]; see also [10] for a introductory account of the problem.

It was proved by Macarini [15], extending an earlier result of Polterovich [19], that if $[\sigma] \neq 0$ there exist non trivial contractible closed orbits of the magnetic flow in a sequence of arbitrarily small energy levels. Kerman [12] proved the same result for magnetic fields given by symplectic forms. This was then recently sharped by Ginzburg and Gürel [11] and finally by Usher [21], where it is proved that when σ is symplectic contractible closed orbits exist for all low energy levels. See also Lu [13] for another interesting approach to the problem in the case of σ symplectic. Perhaps the most general result so far is due to Schlenk [20] where it is shown that for *any* closed 2-form (not necessarily weakly exact), almost every sufficiently small energy level contains a contractible closed orbit.

The aim of this paper is to extend Theorem D of [5] to the weakly exact case. More precisely, we prove the following.

1.1. THEOREM. *Let (M,g) denote a closed connected Riemannian manifold, and let $\sigma \in \Omega^2(M)$ denote a closed 2-form whose pullback to the universal cover $\widetilde{M}$ is exact. Let $c = c(g,\sigma) \in \mathbb{R}\cup\{\infty\}$ denote the Mañé critical value, and let ϕ_t denote the magnetic flow defined by σ. Then:*

(1) *If $c < \infty$ then for all $k > c$ and for each non-trivial homotopy class $\nu \in [\mathbb{T}, M]$, there exists a closed orbit of ϕ_t with energy k such that the projection to M of that orbit belongs to ν.*
(2) *For almost all $k \in (0, c)$, where possibly $c = \infty$, there exists a contractible closed orbit of ϕ_t with energy k.*

The first statement of Theorem 1.1 has, under a mild technical assumption on $\pi_1(M)$, previously been proved by Paternain [18]. We use a completely different method of proof however, which bypasses the need for this additional assumption. For $c(g,\sigma) < \infty$, the second statement of Theorem 1.1 is due to Osuna [17]. It should be emphasized that we believe the main contribution of the present paper is the case $c(g,\sigma) = \infty$ in the second statement.

Remark. We will actually prove a slightly stronger statement than the one stated above; see Proposition 5.8 below for details.

When $\pi_1(M)$ is *amenable* and σ is not exact, we always have $c(g,\sigma) = \infty$ (see for instance [18, Corollary 5.4]). Thus we have the following immediate corollary.

1.2. COROLLARY. *Let (M,g) denote a closed connected Riemannian manifold, and let $\sigma \in \Omega^2(M)$ denote a closed non-exact 2-form whose pullback to the universal cover $\tilde{M}$ is exact. Suppose $\pi_1(M)$ is amenable. Then almost every energy level contains a contractible closed orbit of the magnetic flow defined by σ.*

Let us now give a brief outline of our method of attack. Fix a primitive θ of $\tilde{\sigma}$, and consider the Lagrangian $L : T\tilde{M} \to \mathbb{R}$ defined by

$$L(q,v) := \frac{1}{2}|v|^2 - \theta_q(v).$$

The Euler-Lagrange flow of L is precisely the lifted flow $\tilde{\phi}_t : T\tilde{M} \to T\tilde{M}$ of the magnetic flow $\phi_t : TM \to TM$ (see for example [7]). Recall that the *action* $A(y)$ of the Lagrangian L over an absolutely continuous curve $y : [0,T] \to \tilde{M}$ is given by

$$\begin{aligned} A(y) : &= \int_0^T L(y(t),\dot{y}(t))dt. \\ &= \int_0^T \frac{1}{2}|\dot{y}(t)|^2\,dt - \int_y \theta. \end{aligned}$$

Set

$$A_k(y) := \int_0^T \{L(y(t),\dot{y}(t)) + k\}\,dt = A(y) + kT.$$

A closed orbit of $\tilde{\phi}_t$ with energy k can be realized as a critical point of the functional $y \mapsto A_k(y)$. More precisely, let $\Lambda_{\tilde{M}}$ denote the Hilbert manifold of absolutely continuous curves $x : \mathbb{T} \to \tilde{M}$ and consider $\tilde{S}_k : \Lambda_{\tilde{M}} \times \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\begin{aligned} \tilde{S}_k(x,T) : &= \int_0^1 T \cdot L(x(t),\dot{x}(t)/T)dt + kT \\ &= \int_0^1 \frac{1}{2T}|\dot{x}(t)|^2\,dt + kT - \int_x \theta. \end{aligned}$$

Then the pair (x,T) is a critical point $\tilde{S}_k$ if and only if $y(t) := x(t/T)$ is the projection to $\tilde{M}$ of a closed orbit of $\tilde{\phi}_t$ with energy k (see [8]).

If σ was actually exact then we could define L on TM, instead of just on $T\tilde{M}$. In this case it has been shown in [8] that for $k > c(g,\sigma)$, $\tilde{S}_k$ satisfies the Palais-Smale condition and is bounded below. Standard results from Morse theory [8, Corollary 23] then tell us that $\tilde{S}_k$ admits a global minimum, and this gives us our desired closed orbit. In [5] this was extended to give contractible orbits on almost every energy level below the critical value. Crucially however these results use compactness of M, and hence are not applicable directly in the weakly exact case, since then L is defined only on $T\tilde{M}$.

In the weakly exact case, whilst $\tilde{S}_k$ is not well defined on TM, its differential is. This leads to the key observation of the present work that we can still work directly on Λ_M. More precisely,

we define a functional $S_k : \Lambda_M \times \mathbb{R}^+ \to \mathbb{R}$ with[1] the property that (x,T) is a critical point of S_k if and only if a lift $\tilde{y}$ to $\tilde{M}$ of the curve $y(t) := x(t/T)$ is the projection to $\tilde{M}$ of a flow line of $\tilde{\phi}_t$ with energy k. The functional S_k is given by

$$S_k(x,T) := \int_0^T \frac{1}{2T} \left|\dot{x}(t)\right|^2 dt + kT - \int_{C(x)} \sigma,$$

where $C(x)$ is any cylinder with boundary $x(\mathbb{T}) \cup x_\nu(\mathbb{T})$, where $x_\nu \in \Lambda_M$ is some fixed reference loop in the free homotopy class $\nu \in [\mathbb{T}, M]$ that x belongs to. If $c(g,\sigma) < \infty$, then since σ is weakly exact, the value $\int_{C(x)} \sigma$ is independent of the choice of cylinder $C(x)$ for any curve $x \in \Lambda_M$. In the case $c(g,\sigma) = \infty$, the value $\int_{C(x)} \sigma$ is independent of the choice of cylinder only when x is a contractible loop.

The functional S_k allows one to extend other results previously known only for the exact case to the weakly exact case. For instance, in [16] we will use S_k to establish the short exact sequence [4, 2] between the Rabinowitz Floer homology of a weakly exact twisted cotangent bundle and the singular (co)homology of the free loop space.

Acknowledgment. I would like to thank my Ph.D. adviser Gabriel P. Paternain for many helpful discussions.

2. Preliminaries

The setup.

It will be convenient to view M and $\tilde{M}$ as being embedded isometrically in some $\mathbb{R}^d$ (possible by Nash's theorem). We will be interested in various spaces of absolutely continuous curves.

Firstly, given $q_0, q_1 \in M$ and $T \geq 0$, let $C_M^{\mathrm{ac}}(q_0,q_1;T)$ denote the set of absolutely continuous curves $y : [0,T] \to M$ with $y(0) = q_0$ and $y(T) = q_1$. Let

$$C_M^{\mathrm{ac}}(q_0,q_1) := \bigcup_{T \geq 0} C_M^{\mathrm{ac}}(q_0,q_1;T).$$

We can repeat the construction on $\tilde{M}$ to obtain for $q_0, q_1 \in \tilde{M}$ sets $C_{\tilde{M}}^{\mathrm{ac}}(q_0,q_1;T)$ and $C_{\tilde{M}}^{\mathrm{ac}}(q_0,q_1)$ of curves on $\tilde{M}$.

Next, consider the space

$$W^{1,2}(\mathbb{R}^d) := \left\{ x : I \to \mathbb{R}^m \text{ absolutely continuous } : \int_0^1 \left|\dot{x}(t)\right|^2 dt < \infty \right\},$$

and

$$W^{1,2}(M) := \left\{ x \in W^{1,2}(\mathbb{R}^d) \, : \, x(I) \subseteq M \right\}.$$

with $W^{1,2}(\tilde{M})$ defined similarly. Here $I := [0,1]$, a convention we shall follow throughout the paper.

[1] If $\tilde{\sigma}$ does not admit any bounded primitives S_k is only defined on $\Lambda_0 \times \mathbb{R}^+$, where $\Lambda_0 \subseteq \Lambda_M$ is the subset of contractible loops.

Let $\Lambda_{\mathbb{R}^d} \subseteq W^{1,2}(\mathbb{R}^d)$ denote the set of closed loops of class $W^{1,2}$ on $\mathbb{R}^d$, and let $\Lambda_M := W^{1,2}(M) \cap \Lambda_{\mathbb{R}^d}$. We will think of maps $x \in \Lambda_M$ as maps $x : \mathbb{T} \to M$ (here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, which we shall often identify with S^1). Given a free homotopy class $\nu \in [\mathbb{T}, M]$, let $\Lambda_\nu \subseteq \Lambda_M$ denote the connected component of Λ_M consisting of the loops belonging to ν.

The tangent space to $\Lambda_{\mathbb{R}^d}$ at $x \in \Lambda_{\mathbb{R}^d}$ is given by

$$T_x\Lambda_{\mathbb{R}^d} = \left\{ \xi \in W^{1,2}(\mathbb{R}^d) \,:\, \xi(0) = \xi(1) \right\}.$$

Given $(x, T) \in \Lambda_M \times \mathbb{R}^+$ we thus have

$$T_{(x,T)}(\Lambda_M \times \mathbb{R}^+) = \left\{ (\xi, \psi) \in W^{1,2}(\mathbb{R}^d) \times \mathbb{R} \,:\, \xi(0) = \xi(1) \right\}.$$

Let $\langle \cdot, \cdot \rangle$ denote the standard Euclidean metric. The metric on $W^{1,2}(\mathbb{R}^d)$ we will work with is

$$\langle \xi, \zeta \rangle_{1,2} := \langle \xi(0), \zeta(0) \rangle + \int_0^1 \left\langle \dot{\xi}(t), \dot{\zeta}(t) \right\rangle dt.$$

This defines a metric which we shall denote simply by $\langle \cdot, \cdot \rangle$ on $W^{1,2}(\mathbb{R}^d) \times \mathbb{R}^+$ by

$$\langle (\xi, \psi), (\zeta, \chi) \rangle := \langle \xi, \zeta \rangle_{1,2} + \psi\chi. \tag{2.1}$$

Mañé's critical value.

We now recall the definition of $c(g, \sigma)$, the *critical value* introduced by Mañé in [14], which plays a decisive role in all that follows.

Let us fix a primitive θ of $\tilde{\sigma}$. Given $k \in \mathbb{R}^+$, we define A_k as follows. Let $q_0, q_1 \in \tilde{M}$. Define $A_k : C^{\mathrm{ac}}_{\tilde{M}}(q_0, q_1) \to \mathbb{R}$ by

$$A_k(y) := \int_0^T \frac{1}{2} |\dot{y}(t)|^2 + kT - \int_y \theta.$$

We define *Mañé's action potential* $m_k : \tilde{M} \times \tilde{M} \to \mathbb{R} \cup \{-\infty\}$ by

$$m_k(q_0, q_1) := \inf_{T>0} \inf_{y \in C^{\mathrm{ac}}_{\tilde{M}}(q_0, q_1; T)} A_k(y).$$

Then we have the following result; for a proof see for instance [7, Proposition 2-1.1] for the first five statements, and [3, Appendix A] for a proof of the last statement.

2.1. LEMMA. *Properties of m_k:*

(1) *If $k \le k'$ then $m_k(q_0, q_1) \le m_{k'}(q_0, q_1)$ for all $q_0, q_1 \in \tilde{M}$.*

(2) *For all $k \in \mathbb{R}$ and all $q_0, q_1, q_2 \in \tilde{M}$ we have*

$$m_k(q_0, q_2) \le m_k(q_0, q_1) + m_k(q_1, q_2).$$

(3) *Fix $k \in \mathbb{R}$. Then either $m_k(q_0, q_1) = -\infty$ for all $q_0, q_1 \in \tilde{M}$, or $m_k(q_0, q_1) \in \mathbb{R}$ for all $q_0, q_1 \in \tilde{M}$ and $m_k(q, q) = 0$ for all $q \in \tilde{M}$.*

(4) *If*

$$c(g, \sigma) := \inf \left\{ k \in \mathbb{R} \,:\, m_k(q_0, q_1) \in \mathbb{R} \text{ for all } q_0, q_1 \in \tilde{M} \right\}$$

then $m_{c(g,\sigma)}$ is finite everywhere.

(5) *We can alternatively define $c(g,\sigma)$ as follows:*

$$c(g,\sigma)=\inf_{u\in C^{\infty}(\tilde{M})}\sup_{q\in\tilde{M}}\frac{1}{2}\left|d_q u+\theta_q\right|^2. \tag{2.2}$$

We call the number $c(g,\sigma)$ the *Mañé critical value*. Using (2.2) it is clear that $c(g,\sigma)<\infty$ if and only if θ is bounded, that is if

$$\left\|\theta\right\|_{\infty}:=\sup_{q\in\tilde{M}}\left|\theta_q\right|<\infty. \tag{2.3}$$

The functional S_k.

We will now define a second functional S_k, which will be the main object of study of the present work. In the case $c(g,\sigma)<\infty$, S_k is defined on $\Lambda_M\times\mathbb{R}^+$. For $c(g,\sigma)=\infty$, S_k is only defined on $\Lambda_0\times\mathbb{R}^+$. The following lemma is the key observation required to define S_k. In the statement, $\mathbb{T}^2$ denotes the 2-torus.

2.2. LEMMA. *Suppose $c(g,\sigma)<\infty$. Then for any smooth map $f:\mathbb{T}^2\to M$, $f^*\sigma$ is exact.*

Proof. Consider $G:=f_*(\pi_1(\mathbb{T}^2))\leq\pi_1(M)$. Then G is amenable, since $\pi_1(\mathbb{T}^2)=\mathbb{Z}^2$, which is amenable. Then [18, Lemma 5.3] tells us that since $\left\|\theta\right\|_{\infty}<\infty$ we can replace θ by a G-invariant primitive θ' of $\tilde{\sigma}$, which descends to define a primitive $\theta''\in\Omega^1(\mathbb{T}^2)$ of $f^*\sigma$. □

For each free homotopy class $\nu\in[\mathbb{T},M]$, pick a reference loop $x_\nu\in\Lambda_\nu$. Given any $x\in\Lambda_\nu$, let $C(x)$ denote a cylinder with boundary $x(\mathbb{T})\cup x_\nu(\mathbb{T})$.

Define $S_k:\Lambda_\nu\times\mathbb{R}^+\to\mathbb{R}$ by

$$S_k(x,T):=\int_0^1\frac{1}{2T}\left|\dot{x}(t)\right|^2dt+kT-\int_{C(x)}\sigma,$$

This is well defined because the value of $\int_{C(x)}\sigma$ is *independent of the choice of cylinder*: if $C'(x)$ is another cylinder with boundary $x(\mathbb{T})\cup x_\nu(\mathbb{T})$ then $\mathbb{T}^2(x):=C(x)\cup\overline{C'(x)}$ is a torus (where $\overline{C'(x)}$ denotes the cylinder $C'(x)$ taken with the opposite orientation), and $\int_{\mathbb{T}^2(x)}\sigma=0$ as $\sigma|_{\mathbb{T}^2(x)}$ is exact by the previous lemma.

If $c(g,\sigma)=\infty$ then we cannot define S_k on all of $\Lambda_M\times\mathbb{R}^+$, since in this case Lemma 2.2 fails. It is however well defined on $\Lambda_0\times\mathbb{R}^+$. In order to see why, let us consider the case of contractible loops when $c(g,\sigma)<\infty$ again. If $x:\mathbb{T}\to M$ is contractible and $\boldsymbol{x}:D^2\to M$ denotes a capping disc, so that $\boldsymbol{x}|_{\partial D^2}=x$, then it is easy to see that

$$\int_{C(x)}\sigma=\int_{D^2}\boldsymbol{x}^*\sigma; \tag{2.4}$$

note that the right-hand side is (as it should be) independent of the choice of capping disc $\boldsymbol{x}$, and depends only on x and σ, since $\sigma|_{\pi_2(M)}=0$. Moreover the right-hand side is well defined and depends only on x and σ even when $c(g,\sigma)=\infty$. Thus it makes sense to *define* $S_k|_{\Lambda_0\times\mathbb{R}^+}$ by

$$S_k(x,T)=\int_0^1\frac{1}{2T}\left|\dot{x}(t)\right|^2dt+kT-\int_{D^2}\boldsymbol{x}^*\sigma;$$

this is consistent with the previous definition of $S_k|_{\Lambda_0\times\mathbb{R}^+}$ when $c(g,\sigma)<\infty$.

Next we will explicitly calculate the derivative of S_k. Let (x_s,T_s) be a variation of (x,T), with $\xi(t):=\frac{\partial}{\partial s}\big|_{s=0}x_s(t)$ and $\psi:=\frac{\partial}{\partial s}\big|_{s=0}T_s$. Write E_q and E_v for $\frac{\partial E}{\partial q}$ and $\frac{\partial E}{\partial v}$ respectively. Then an easy calculation in local coordinates show that the first variation (i.e. the Gateaux derivative) of S_k at (ξ,ψ), that is, $\frac{\partial}{\partial s}\big|_{s=0}S_k(x_s,T_s)$ is given by:

$$\begin{aligned}\frac{\partial}{\partial s}\Big|_{s=0}S_k(x_s,T_s)&=\psi\int_0^1\{k-E(x(t),\dot{x}(t)/T)\}\,dt+\int_0^1\sigma_{x(t)}(\xi(t),\dot{x}(t))dt\\ &+\int_0^1\Big\{T\cdot E_q(x(t),\dot{x}(t)/T)\cdot\xi(t)+E_v(x(t),\dot{x}(t)/T)\cdot\dot{\xi}(t)\Big\}\,dt.\end{aligned}\tag{2.5}$$

We claim now that S_k is differentiable with respect to the canonical Hilbert manifold structure of $\Lambda_\nu\times\mathbb{R}^+$ (i.e. S_k is Fréchet differentiable). In fact, S_k is of class C^2. For this we quote the fact that

$$(x,T)\mapsto\int_0^1\frac{1}{2T}\,|\dot{x}(t)|^2\,dt+kT$$

is of class C^2 (see for instance [1]) and thus is remains to check that $x\mapsto\int_{C(x)}\sigma$ is differentiable. This can be checked directly. It thus follows that the first variation $\frac{\partial}{\partial s}\big|_{s=0}S_k(x_s,T_s)$ is actually equal to the (Fréchet) derivative $d_{(x,T)}S_k(\xi,\psi)$.

Finally let us note that

$$\frac{\partial}{\partial T}S_k(x,T)=\frac{1}{T}\int_0^T\{k-2E(y,\dot{y})\}dt,\tag{2.6}$$

where $y(t):=x(t/T)$.

Relating S_k and A_k.

Next, if (x,T) is a critical point of S_k then $y(t):=x(t/T)$ satisfies

$$\int_0^T\left\{E_q(y,\dot{y})-\frac{d}{dt}E_v(y,\dot{y})\right\}\zeta dt-\frac{1}{T}\int_0^T\sigma_y(\zeta,\dot{y})dt=0,$$

where $\zeta(t)=\xi(t/T)$. Since this holds for all variations ζ, this implies that if $\tilde{y}:[0,T]\to\tilde{M}$ is a lift of y then $\tilde{y}$ satisfies the Euler-Lagrange equations for L, that is,

$$L_q(\tilde{y},\dot{\tilde{y}})-\frac{d}{dt}L_v(\tilde{y},\dot{\tilde{y}})=0.$$

Thus $\tilde{y}$ is the lift to $\tilde{M}$ of the projection to M of an orbit of ϕ_t, and we have the following result.

2.3. Corollary. *Let $x\in\Lambda_M$ and $\tilde{x}$ denote a lift of x to $\tilde{M}$. Let $T\in\mathbb{R}^+$. Define $\tilde{y}(t):=\tilde{x}(t/T)$. Then the following are equivalent:*

(1) *The pair (x,T) is a critical point of S_k,*
(2) *$\tilde{y}$ is a critical point of A_k.*

Thus the pair $(x,T)\in\Lambda_M\times\mathbb{R}^+$ is a critical point of S_k if and only if $t\mapsto x(t/T)$ is the projection to M of a closed orbit of ϕ_t.

In order to specify the lifts we work with, let us fix a lift $\tilde{x}_\nu : I\to\tilde{M}$ of x_ν for each $\nu\in[\mathbb{T},M]$. We remark here that throughout the paper given any two paths y,y' such that the end point of y is the start point of y', the path $y*y'$ is the path obtained by first going along y and then going along y'. Similarly the path y^{-1} is the path obtained by going along y backwards.

Suppose now that $c(g,\sigma)<\infty$. Fix a free homotopy class $\nu\in[\mathbb{T},M]$ (which could be the trivial free homotopy class). Let $x\in\Lambda_\nu$, and let x_s denote a free homotopy from $x_0=x$ to $x_1=x_\nu$. Let $z(s):=x_s(0)$. Let $\tilde{x}_s$ denote the unique homotopy of curves on $\tilde{M}$ that projects down onto x_s and satisfies with $\tilde{x}_1(t)=\tilde{x}_\nu(t)$. Let $\tilde{x}(t):=\tilde{x}_0(t)$, $\tilde{z}_0(s):=\tilde{x}_s(0)$ and $\tilde{z}_1(s):=\tilde{x}_s(1)$.

Now observe that if $R\subseteq\tilde{M}$ denotes the rectangle $R=\operatorname{im}\tilde{x}_s$ then we have

$$\int_{C(x)}\sigma=\int_R\tilde{\sigma}=\int_R d\theta=\int_{\partial R}\theta=\int_{\tilde{x}*\tilde{z}_1*\tilde{x}_\nu^{-1}*\tilde{z}_0^{-1}}\theta.$$

Let $\varphi\in\pi_1(M)$ denotes the unique covering transformation taking $\tilde{z}_0$ to $\tilde{z}_1$. Since $\langle\varphi\rangle\le\pi_1(M)$ is an amenable subgroup, [18, Lemma 5.3] allows us to assume that without loss of generality, θ is φ-invariant. Thus

$$\int_{\tilde{z}_0^{-1}}\theta+\int_{\tilde{z}_1}\theta=0.$$

It thus follows that

$$\int_{C(x)}\sigma=\int_{\tilde{x}}\theta+\int_{\tilde{x}_\nu^{-1}}\theta.$$

Set

$$a_\nu:=\int_{\tilde{x}_\nu^{-1}}\theta.$$

We conclude:

$$\int_{C(x)}\sigma=\int_{\tilde{x}}\theta+a_\nu.$$

This computation shows if $c(g,\sigma)<\infty$ then for any $(x,T)\in\Lambda_\nu\times\mathbb{R}^+$, if $\tilde{x}$ is any lift of x and $\tilde{y}(t):=\tilde{x}(t/T)$ then we have

$$S_k(x,T)=A_k(\tilde{y})+a_\nu. \tag{2.7}$$

For the case $\nu=0\in[\mathbb{T},M]$ the trivial free homotopy class, we may choose above the curve x_0 to be a constant map, from which it is easy to see that $a_0=0$. In particular, if $(x,T)\in\Lambda_0\times\mathbb{R}^+$ and $\tilde{y}$ is defined as before then

$$S_k(x,T)=A_k(\tilde{y}). \tag{2.8}$$

Finally, if $c(g,\sigma)=\infty$, S_k is only defined on $\Lambda_0\times\mathbb{R}^+$, and it is clear that (2.8) still holds.

3. THE PALAIS-SMALE CONDITION

Let $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ be a Riemannian Hilbert manifold, and let $S : \mathcal{M} \to \mathbb{R}$ be of class C^1.

3.1. DEFINITION. *We say that S satisfies the* Palais-Smale condition *if every sequence $(x_n) \subseteq \mathcal{M}$ such that $\|d_{x_n}S\| \to 0$ as $n \to \infty$ and $\sup_n S(x_n) < \infty$ admits a convergent subsequence. We say that S satisfies the* Palais-Smale condition at the level $\mu \in \mathbb{R}$ *if every sequence $(x_n) \subseteq \mathcal{M}$ with $\|d_{x_n}S\| \to 0$ as $n \to \infty$ and $S(x_n) \to \mu$ admits a convergent subsequence.*

In this section we will prove the following result about when the functional S_k satisfies the Palais-Smale condition. This result is adapted from [5, Proposition 3.8, Proposition 3.12] We will first consider only the case where $c(g,\sigma) < \infty$ (see Proposition 3.7 below for the case $c(g,\sigma) = \infty$). In the statement of the theorem, $\|\cdot\|$ denotes the operator norm with respect to the metric $\langle\cdot,\cdot\rangle$.

3.2. THEOREM. *Suppose $c(g,\sigma) < \infty$. Let $A, B, k \in \mathbb{R}^+$, and suppose $(x_n, T_n) \subseteq \Lambda_M \times \mathbb{R}^+$ satisfies:*

$$\sup_n |S_k(x_n,T_n)| \leq A, \quad \sup_n T_n \leq B, \quad \left\|d_{(x_n,T_n)}S_k\right\| < \frac{1}{n}.$$

Then if:

(1) *If $\liminf T_n > 0$ then passing to a subsequence if necessary the sequence (x_n, T_n) is convergent in the $W^{1,2}$-topology.*
(2) *If $\liminf T_n = 0$ and the x_n are all contractible, passing to a subsequence if necessary it holds that $S_k(x_n,T_n) \to 0$.*

Before proving the theorem, let us now fix some notation that we will use throughout this section, as well as implicitly throughout the rest of the paper. Given a sequence $(x_n, T_n) \subseteq \Lambda_M \times \mathbb{R}^+$, let $y_n : [0, T_n] \to M$ be defined by $y_n(t) := x_n(t/T_n)$. Define:

$$l_n := \int_0^1 |\dot{x}_n(t)|\, dt;$$

$$e_n := \int_0^1 \frac{1}{2T_n} |\dot{x}_n(t)|^2\, dt.$$

Note that l_n is the length of y_n and e_n is the energy of y_n. The Cauchy-Schwarz inequality implies

$$l_n^2 \leq 2T_n e_n. \tag{3.1}$$

Suppose now $c(g,\sigma) < \infty$. Since $\|\theta\|_\infty < \infty$, there exist constants $b_1, b_2 \in \mathbb{R}^+$ such that

$$L(q,v) \geq b_1 |v|^2 - b_2 \tag{3.2}$$

for all $(q,v) \in T\tilde{M}$.

Given $A, B, k \in \mathbb{R}^+$ and a free homotopy class $\nu \in [\mathbb{T}, M]$, let $\mathbb{D}(A,B,k,\nu) \subseteq \Lambda_M \times \mathbb{R}^+$ denote the set of pairs (x,T) such that $x \in \Lambda_\nu$, $S_k(x,T) \leq A$ and $T \leq B$.

Proof of Theorem 3.2.

We begin with three preparatory lemmata.

3.3. LEMMA. *Suppose* $c(g,\sigma)<\infty$*. Let* $(x_n,T_n)\subseteq\mathbb{D}(A,B,k,\nu)$*. Then if*

$$b(A,B,\nu):=\frac{A+b_2B+|a_\nu|}{2b_1}$$

it holds that

$$e_n\leq b(A,B,\nu)$$

for all $n\in\mathbb{N}$*.*

Proof. We have by (2.7) and (3.2) that

$$\begin{aligned}A&\geq S_k(x_n,T_n)\\&=A_k(\tilde{y}_n)-a_\nu\\&\geq 2b_1e_n-b_2T_n+kT_n-a_\nu,\end{aligned}$$

and thus

$$e_n\leq\frac{A+b_2T_n-kT_n+|a_\nu|}{2b_1}\leq\frac{A+b_2B+|a_\nu|}{2b_1}.$$

□

3.4. LEMMA. *Suppose* $c(g,\sigma)<\infty$*, and suppose* $(x_n)\subseteq\Lambda_0$ *are such that* $l_n\to0$*. Then* $\int_{C(x_n)}\sigma\to0$*.*

Proof. Let $\boldsymbol{x}_n:D^2\to M$ denote a capping disc for x_n, so $\boldsymbol{x}_n|_{\partial D^2}=x_n$ and $\int_{C(x_n)}\sigma=\int_{D^2}\boldsymbol{x}_n^*\sigma$ (as in (2.4)).

Let $\tilde{\boldsymbol{x}}_n:D^2\to\tilde{M}$ denote a lift of $\boldsymbol{x}_n$ to $\tilde{M}$. Then

$$\left|\int_{D^2}\boldsymbol{x}_n^*\sigma\right|=\left|\int_{D^2}\tilde{\boldsymbol{x}}_n^*(d\theta)\right|=\left|\int_{\tilde{x}_n}\theta\right|\leq\|\theta\|_\infty l_n\to0.$$

□

We now reduce the first statement of Theorem 3.2 to the following simpler situation:

3.5. LEMMA. *Suppose* $c(g,\sigma)<\infty$ *and* $(x_n,T_n)\in\mathbb{D}(A,B,k,\nu)$ *with* $\liminf T_n>0$*. Passing to a subsequence we may assume that there exists* $x\in\Lambda_\nu$ *such that the* x_n *converge to* x *in the* C^0*-topology.*

Proof. Firstly by compactness of M, passing to a subsequence if necessary we may assume there exists $q\in M$ and $T\in\mathbb{R}^+$ such that $\lim_{n\to\infty}x_n(0)=x_n(1)=q$ and $\lim_{n\to\infty}T_n=T$. Consider g-geodesics $c_n:I\to M$ such that $c_n(0)=q$ and $c_n(1)=x_n(0)$. By passing to a subsequence we may assume that $\mathrm{dist}_g(x_n(0),q)<1$, and thus we have $|\dot{c}_n|\leq1$. Now consider the curves $w_n:[0,T_n+2]\to M$ defined by

$$w_n(t)=c_n*y_n*c_n^{-1},$$

and $z_n : \mathbb{T} \to M$ defined by

$$z_n(t) = w_n(t/T_n + 2).$$

Thus $z_n(0) = z_n(1) = q$, and $(z_n) \subseteq \Lambda_\nu$.

Given $0 \le t_1 < t_2 < T_n + 2$,

$$\mathrm{dist}_g(w_n(t_1), w_n(t_2)) \le \int_{t_1}^{t_2} |\dot{w}_n(t)|\, dt \le \sqrt{2}\, |t_2 - t_1|^{1/2} \left(\int_0^{T_n+2} \frac{1}{2} |\dot{w}_n(t)|^2 \, dt \right)^{1/2}.$$

By Lemma 3.3 we have

$$\begin{aligned} \int_0^{T_n+2} \frac{1}{2} |\dot{w}_n(t)|^2 \, dt &= \int_0^1 \frac{1}{2} |\dot{c}_n(t)|^2 \, dt + e_n + \int_0^1 \frac{1}{2} \left|\dot{c}_n^{-1}(t)\right|^2 dt \\ &\le 1 + b(A, B, \nu), \end{aligned}$$

and thus

$$\mathrm{dist}_g(w_n(t_1), w_n(t_2)) \le \sqrt{2}\, |t_2 - t_1|^{1/2} \,(1 + b(A, B, \nu))^{1/2}.$$

Hence the family (w_n) is equicontinuous. The Arzelá-Ascoli theorem then completes the proof. $\square$

We will now prove Theorem 3.2.

Proof. (of Theorem 3.2)

We begin by proving the first statement of the theorem. This part of the proof is very similar to the proof of [8, Theorem B]. Suppose $(x_n, T_n) \subseteq \mathbb{D}(A, B, k, \nu)$ with $\liminf T_n > 0$. By the previous lemma, after passing to a subsequence if necessary, we may assume that (x_n, T_n) converges in the C^0-topology to some (x, T) where $T > 0$.

Without loss of generality, let us assume that the limit curve x is contained in a single chart U (otherwise simply repeat these arguments finitely many times). Then after passing possibly to another subsequence, we may assume that the x_n are all contained in U as well. There exists a constant $b_3 \in \mathbb{R}^+$ such that in the coordinates on U,

$$b_3 := \sup_{q \in U, v \in T_q M} \frac{|E_q(q, v)|}{1 + |v|^2} < \infty. \tag{3.3}$$

Write $z_n(t) := \frac{1}{T_n} x_n(t)$. By Lemma 3.3 we can find a constant $R > 0$ such that

$$|x_n|_{1,2} \le R, \quad |z_n|_{1,2} \le R.$$

Now since $\left\| d_{(x_n, T_n)} S_k \right\| \to 0$ as $n \to \infty$, given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every (ξ, ψ) satisfying $|(\xi, \psi)| \le 2R$ and $n, m \ge N$ we have

$$\left| d_{(x_n, T_n)} S_k(\xi, \psi) - d_{(x_m, T_m)} S_k(\xi, \psi) \right| < \varepsilon.$$

Take $\xi = x_n - x_m$ and $\psi = 0$ and use (2.5) to discover:

$$\begin{aligned}(3.4)\quad \Big| \int_0^1 \left\{ T_n \cdot E_q(x_n, \dot{z}_n) - T_m \cdot E_q(x_m, \dot{z}_m) \right\} (x_n - x_m) dt \\ + \int_0^1 \left\{ E_v(x_n, \dot{z}_n) - E_v(x_m, \dot{z}_m) \right\} (\dot{x}_n - \dot{x}_m) dt \\ + \int_0^1 \sigma_{x_n}(\dot{x}_n, \dot{x}_m) - \sigma_{x_m}(\dot{x}_n, \dot{x}_m) dt \Big| < \varepsilon.\end{aligned}$$

Here we are using the canonical parallel transport available to us on Euclidean spaces to view $\dot{x}_n - \dot{x}_m$ as a tangent vector in any tangent space of our choosing. Using (3.3) we can bound the first integral as follows:

$$\left| \int_0^1 \left\{ T_n \cdot E_q(x_n, \dot{z}_n) - T_m \cdot E_q(x_m, \dot{z}_m) \right\} (x_n - x_m) dt \right| \leq (2Bb_3 + 2b_3 b(A, B, \nu)) \left\| x_n - x_m \right\|_\infty .$$

Let us write $\sigma|_U$ in local coordinates as $\sigma = \sigma_{ij} dq^i \wedge dq^j$, where $\sigma_{ij} \in C^\infty(U, \mathbb{R})$. Then since

$$|\sigma_{ij}(x_n(t)) - \sigma_{ij}(x_m(t))| \to 0 \quad \text{as } n, m \to \infty, \text{ uniformly in } t,$$

and

$$\int_0^1 |\dot{x}_n| \, |\dot{x}_m| \, dt \leq 2\sqrt{T_n T_m e_n e_m}$$

is bounded, it follows that for n, m large the third integral is small. Thus the second integral must also be small for large n, m. Since

$$|v - v'|^2 = (E_v(q, v) - E_v(q', v')) \cdot (v - v'),$$

we have

$$\int_0^1 |\dot{z}_n - \dot{z}_m|^2 \, dt \leq \int_0^1 \left\{ E_v(x_n, \dot{z}_n) - E_v(x_m, \dot{z}_m) \right\} (\dot{z}_n - \dot{z}_m) dt,$$

and hence the fact that the second integral in (3.4) is small for large n, m implies that the sequence (z_n), and hence the sequence (x_n), converges in the $W^{1,2}$-topology. This completes the proof of the first statement of Theorem 3.2.

We now prove the second statement of Theorem 3.2. This part of the proof follows the proof of [5, Theorem 3.8] very closely. Assume $(x_n, T_n) \subseteq \mathbb{D}(A, B, k, 0)$ (where $0 \in [\mathbb{T}, M]$ denotes the trivial free homotopy class) and that $\liminf T_n = 0$. Passing to a subsequence we may assume that $T_n \to 0$. It suffices to show that passing to a subsequence we have $e_n \to 0$. Then

$$S_k(x_n, T_n) = e_n + kT_n - \int_{C(x_n)} \sigma \to 0$$

by Lemma 3.4.

We know that e_n remains bounded by Lemma 3.3. Since $T_n \to 0$, (3.1) implies that $l_n \to 0$. Thus as before we may assume that all the curves x_n take their image in the domain of some chart U on M. Thus for the remainder of the proof we work in coordinates as if $M = \mathbb{R}^{\dim M}$.

Let $\xi_n(t) := x_n(t) - x_n(0)$ so that $\xi_n(0) = \xi_n(1) = 0$. Then $(\xi_n, 0) \in T_{(x_n,T_n)}(\Lambda_{\mathbb{R}^{\dim M}} \times \mathbb{R}^+)$. Let also $\zeta_n(t) := \xi_n(t/T_n)$, so that $\dot{\zeta}_n(t) = \dot{y}_n(t)$. Then

$$\begin{aligned}\left|d_{(x_n,T_n)}S_k(\xi_n, 0)\right| &\leq \frac{1}{n}\left(T_n \int_0^{T_n} \left|\dot{\zeta}_n(t)\right|^2 dt\right)^{1/2} \\ &\leq \frac{1}{n}\sqrt{2T_n e_n}.\end{aligned}$$

Using (2.5) we have

$$\begin{aligned}d_{(x_n,T_n)}S_k(\xi_n, 0) = &\int_0^{T_n} \left\{E_q(y_n, \dot{y}_n) \cdot \zeta_n + E_v(y_n, \dot{y}_n) \cdot \dot{\zeta}_n(t)\right\} dt \\ &+ \int_0^1 \sigma_{x_n(t)}(\xi_n(t), \dot{x}_n(t))dt.\end{aligned}$$

There exists $b_4 \in \mathbb{R}^+$ such that

$$\left|\int_0^1 \sigma_{x_n(t)}(\xi_n(t), \dot{x}_n(t))dt\right| \leq b_4 \int_0^1 |\xi_n(t)|\, |\dot{x}_n(t)|\, dt \leq b_4 l_n^2.$$

Thus using (3.3) and the fact that

$$E_v(q, v) \cdot \xi = \langle v, \xi\rangle$$

we have

$$\begin{aligned}d_{(x_n,T_n)}S_k(\xi_n, 0) &\geq -b_3 \int_0^{T_n} \left(1 + |\dot{y}_n(t)|^2\right) |y_n(t) - y_n(0)|\, dt + 2e_n - b_4 l_n^2 \\ &\geq -b_3 l_n(T_n + 2e_n) + 2e_n - b_4 l_n^2.\end{aligned}$$

Putting this together and dividing through by $\sqrt{T_n}$ we have

$$-b_3 l_n\sqrt{T_n} - 2b_3\frac{e_n l_n}{\sqrt{T_n}} + 2\frac{e_n}{\sqrt{T_n}} - b_4\frac{l_n^2}{\sqrt{T_n}} \leq \frac{1}{n}\sqrt{2e_n}.$$

By (3.1), we have

$$\lim_{n\to\infty} \frac{l_n^2}{\sqrt{T_n}} = 0, \quad \lim_{n\to\infty} \frac{l_n}{\sqrt{T_n}} \quad \text{bounded;}$$

thus we must have

$$\lim_{n\to\infty} \frac{e_n}{\sqrt{T_n}} \quad \text{bounded,}$$

and this can happen if and only if $e_n \to 0$. This completes the proof of the second statement of Theorem 3.2. $\square$

We now wish to study the case where $c(g, \sigma) = \infty$. Recall in this case S_k is only defined on $\Lambda_0 \times \mathbb{R}^+$. In order for a result similar to the above theorem to hold in the unbounded setting, we must restrict to a subset of $\Lambda_0 \times \mathbb{R}^+$.

3.6. DEFINITION. *Suppose $K \subseteq \tilde{M}$ is compact. Define $\Lambda_0^K \subseteq \Lambda_0$ to be the set of loops $x \in \Lambda_0$ such that there exists a lift $\tilde{x} : \mathbb{T} \to \tilde{M}$ of x such that $\tilde{x}(\mathbb{T}) \subseteq K$.*

Here is the extension of Theorem 3.2 to the case $c(g,\sigma)=\infty$.

3.7. PROPOSITION. *Suppose that* $c(g,\sigma)=\infty$. *Let* $A,B,k\in\mathbb{R}^+$ *and take* $K\subseteq\tilde{M}$ *compact. Suppose* $(x_n,T_n)\subseteq\Lambda_0^K\times\mathbb{R}^+$ *satisfy:*

$$\sup_n|S_k(x_n,T_n)|\leq A,\quad \sup_n T_n\leq B,\quad \left\|d_{(x_n,T_n)}S_k\right\|<\frac{1}{n}.$$

Then:

1. *If* $\liminf T_n>0$ *then passing to a subsequence if necessary the sequence* (x_n,T_n) *is convergent in the* $W^{1,2}$*-topology.*
2. *If* $\liminf T_n=0$, *passing to a subsequence if necessary it holds that* $S_k(x_n,T_n)\to 0$.

Proof. The proof proceeds exactly as before, since any primitive θ of $\tilde{\sigma}$ is necessarily bounded on K. □

4. SUPERCRITICAL ENERGY LEVELS: THE CASE $k>c(g,\sigma)$

In this section we assume $c(g,\sigma)<\infty$, and study *supercritical energies* $k>c(g,\sigma)$. We aim to prove the first statement of Theorem 1.1. The key fact we will use is the following result. As before, let $(\mathcal{M},\langle\cdot,\cdot\rangle)$ be a Riemannian Hilbert manifold, and let $S:\mathcal{M}\to\mathbb{R}$ be of class C^1.

4.1. PROPOSITION. *Suppose* S *is bounded below, satisfies the Palais-Smale condition and for every* $A\in\mathbb{R}^+$ *the set* $\{x\in\mathcal{M}\,:\,S(x)\leq A\}$ *is complete. Then* S *has a global minimum.*

A proof may be found in [8, Corollary 23]. Fix a non-trivial free homotopy class $\nu\in[\mathbb{T},M]$. The aim of this section is to verify that for $k>c(g,\sigma)$, the functional S_k on the Hilbert manifold $\Lambda_\nu\times\mathbb{R}^+$ satisfies the hypotheses of Proposition 4.1. For then the global minimum whose existence Theorem 4.1 guarantees is our desired closed orbit of energy k.

The first step then is the following lemma, whose proof only requires $k\geq c(g,\sigma)$, and works for any free homotopy class $\nu\in[\mathbb{T},M]$.

4.2. LEMMA. *Let* $k\geq c(g,\sigma)$. *Then* $S_k|_{\Lambda_\nu\times\mathbb{R}^+}$ *is bounded below.*

Proof. The argument begins by replicating an argument seen earlier in Section 2. Fix a free homotopy class $\nu\in[\mathbb{T},M]$ (which could be the trivial free homotopy class). Let $(x,T)\in\Lambda_\nu\times\mathbb{R}^+$, and let x_s denote a free homotopy from $x_0=x$ to $x_1=x_\nu$. Let $z(s):=x_s(0)$. Lift x_s to a homotopy $\tilde{x}_s$ in $\tilde{M}$ with $\tilde{x}_1(t)=\tilde{x}_\nu(t)$, and let $\tilde{x}(t):=\tilde{x}_0(t)$, $\tilde{z}_0(s)=\tilde{x}_s(0)$ and $\tilde{z}_1(s)=\tilde{x}_s(1)$.

Now observe that if $R\subseteq\tilde{M}$ denotes the rectangle $R=\operatorname{im}\tilde{x}_s$ then we have

$$\int_{C(x)}\sigma=\int_R\tilde{\sigma}=\int_R d\theta=\int_{\partial R}\theta=\int_{\tilde{x}*\tilde{z}_1*\tilde{x}_\nu^{-1}*\tilde{z}_0^{-1}}\theta.$$

Suppose $\varphi\in\pi_1(M)$ denotes the unique covering transformation taking $\tilde{z}_0$ to $\tilde{z}_1$. Since $\langle\varphi\rangle\leq\pi_1(M)$ is an amenable subgroup, [18, Lemma 5.3] allows us to assume that without loss of generality, θ is φ-invariant. Thus

$$\int_{\tilde{z}_0^{-1}}\theta+\int_{\tilde{z}_1}\theta=0.$$

It thus follows that

$$\int_{C(x)} \sigma = \int_{\tilde{x}} \theta + \int_{\tilde{x}_\nu^{-1}} \theta. \tag{4.1}$$

Let $\tilde{x}_n := \varphi^n \tilde{x}$, and use similar notations for $\tilde{z}_n$ and $\tilde{x}_{\nu,n}$. Let $\tilde{y}_n := \tilde{x}_n(t/T)$, so $\tilde{y}_n : [0,T] \to \tilde{M}$. Then for any $n \in \mathbb{N}$ we can consider the closed loop $u_n : [0, T_n] \to \tilde{M}$ defined by

$$u_n = \tilde{y}_0 * \tilde{y}_1 * \cdots * \tilde{y}_n * \tilde{z}_{n+1} * \tilde{x}_{\nu,n}^{-1} * \cdots * \tilde{x}_{\nu,1}^{-1} * \tilde{x}_\nu^{-1} * \tilde{z}_0^{-1},$$

where

$$T_n := (n+1)T + 1 + (n+1) + 1.$$

We have

$$\begin{aligned} A_k(u_n) = (n+1) &\left\{ \int_0^T \frac{1}{2} \left| \dot{\tilde{y}}(t) \right|^2 dt + \int_0^1 \frac{1}{2} \left| \dot{\tilde{x}}_\nu^{-1} \right|^2 dt - \int_{\tilde{y}_0} \theta - \int_{\tilde{x}_\nu^{-1}} \theta \right\} \\ &+ \int_0^1 \frac{1}{2} \left| \dot{\tilde{z}}_1(t) \right|^2 dt + \int_0^1 \frac{1}{2} \left| \dot{\tilde{z}}_0^{-1}(t) \right|^2 dt + kT_n. \end{aligned}$$

Now if $k \geq c(g,\sigma)$ then by definition of $c(g,\sigma)$ we have $A_k(u_n) \geq 0$. We thus obtain

$$\begin{aligned} 0 \leq &\int_0^T \frac{1}{2} \left| \dot{\tilde{y}}_0(t) \right|^2 dt + \int_0^1 \frac{1}{2} \left| \dot{\tilde{x}}_\nu^{-1} \right|^2 dt - \int_{\tilde{y}_0} \theta - \int_{\tilde{x}_\nu^{-1}} \theta + \frac{kT_n}{n+1} \\ &+ \frac{1}{n+1} \left(\int_0^1 \left| \dot{\tilde{z}}_1(t) \right|^2 dt + \int_0^1 \left| \dot{\tilde{z}}_0^{-1}(t) \right|^2 dt \right). \end{aligned}$$

Letting $n \to \infty$ and substituting for the terms with $\tilde{y}_0$ we obtain

$$\int_0^1 \frac{1}{2T} \left| \dot{x}(t) \right|^2 dt + \int_0^1 \frac{1}{2} \left| \dot{x}_\nu^{-1} \right|^2 dt - \int_{\tilde{x}} \theta - a_\nu + k(T+1) \geq 0. \tag{4.2}$$

Now

$$\begin{aligned} S_k(x,T) &= \int_0^1 \frac{1}{2T} \left| \dot{x}(t) \right|^2 dt + kT - \int_{C(x)} \sigma \\ &= \int_0^1 \frac{1}{2T} \left| \dot{x}(t) \right|^2 dt + kT - \int_{\tilde{x}} \theta - a_\nu, \end{aligned}$$

and hence by (4.1) and (4.2),

$$S_k(x,T) + \int_0^1 \frac{1}{2} \left| \dot{\tilde{x}}_\nu(t) \right|^2 dt + k \geq 0,$$

that is,

$$S_k(x,T) \geq -\int_0^1 \frac{1}{2} \left| \dot{\tilde{x}}_\nu(t) \right|^2 dt - k > -\infty,$$

which completes the proof. □

Let us set

$$i_{k,\nu} := \inf_{(x,T)\in\Lambda_\nu\times\mathbb{R}^+} S_k(x,T),$$

so that the lemma tells us $i_{k,\nu} > -\infty$ for $k \geq c(g,\sigma)$.

The next lemma implies that $\{S_k|_{\Lambda_\nu\times\mathbb{R}^+} \leq A\}$ is complete for any $A \geq 0$.

4.3. LEMMA. *Suppose* $c(g,\sigma) < \infty$. *Let* $\nu \in [\mathbb{T}, M]$ *be a non-trivial free homotopy class and* $A \in \mathbb{R}^+$. *There exists* $T_0 = T_0(A,k,\nu) \in \mathbb{R}^+$ *such that if* $(x,T) \in \mathbb{D}(A,\infty,k,\nu)$ *then* $T \geq T_0$.

Proof. Let $\tilde{x}$ denote an admissable lift of x and let $\tilde{y} : [0,T] \to \tilde{M}$ be the curve $t \mapsto \tilde{x}(t/T)$. Using (2.7) and (3.2) we compute that

$$\begin{aligned} A &\geq S_k(x,T) \\ &= A_k(\tilde{y}) + a_\nu \\ &\geq \frac{b_1}{T}\int_0^1 \left|\dot{\tilde{x}}\right|^2 dt - (k-b_2)T + a_\nu \\ &\geq \frac{b_1}{T} l(\nu) - (k-b_2)T + a_\nu, \end{aligned}$$

where

$$l(\nu) := \inf\left\{\int_0^1 |\dot{x}(t)|\, dt \,:\, x \in \Lambda_\nu\right\}.$$

Since M is closed and ν is a non-trivial free homotopy class, we have $l(\nu) > 0$, which implies the thesis of the lemma. $\square$

Proof of the first statement of Theorem 1.1.

Take $k > c(g,\sigma)$, and fix is a non-trivial free homotopy class $\nu \in [\mathbb{T}, M]$. Let $(x_n, T_n) \subseteq \mathbb{D}(A,\infty,k,\nu)$. We want to show that (x_n,T_n) admits a convergent subsequence in the $W^{1,2}$-topology. In view of Theorem 3.2, it suffices to show that there exists $B > 0$ such that $\mathbb{D}(A,B,k,\nu)$ and that $\liminf T_n > 0$.

4.4. LEMMA. *The sequence* (T_n) *is bounded above and bounded away from zero.*

Proof. First we claim that (T_n) is bounded. Indeed, if $c = c(g,\sigma)$,

$$\begin{aligned} A &\geq S_k(x_n,T_n) \\ &= S_c(x_n,T_n) + (k-c)T_n \\ &\geq i_{c,\nu} + (k-c)T_n, \end{aligned}$$

and thus (T_n) is bounded, say $T_n \leq B$ for all n, where $B \in \mathbb{R}^+$. Passing to a subsequence we may assume that if $T := \liminf T_n$ then $T_n \to T$. It remains to check $T > 0$. From (3.1) and Lemma 3.3 if $T = 0$ then $l_n \to 0$. But this is a contradiction as $l_n > l(\nu) > 0$ (see the proof of the previous lemma). $\square$

5. SUBCRITICAL ENERGY LEVELS: THE CASE $k < c(g,\sigma)$

In this section we drop the assumption that $c(g,\sigma) < \infty$, and study *subcritical energies* $k < c(g,\sigma)$.

Mountain pass geometry.

As before, let $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ be a Riemannian Hilbert manifold and $S : \mathcal{M} \to \mathbb{R}$ a function of class C^2. Let Φ_s denote the (local) flow of $-\nabla S$. Define $\alpha : \mathcal{M} \to \mathbb{R}^+ \cup \{\infty\}$ by

$$\alpha(x) := \sup\{r > 0 \,:\, s \mapsto \Phi_s(x) \text{ is defined on } [0,r]\}.$$

An *admissible time* is a differentiable function $\tau : \mathcal{M} \to \mathbb{R}$ such that

$$0 \leq \tau(x) < \alpha(x) \quad \text{for all } x \in \mathcal{M}.$$

Let $\mathcal{F}$ denote a family of subsets of $\mathcal{M}$, and define

$$\mu := \inf_{F \in \mathcal{F}} \sup_{x \in F} S(x).$$

Suppose that $\mu \in \mathbb{R}$. We say that $\mathcal{F}$ is S*-forward invariant* if the following holds: if τ is an admissible time such that $\tau(x) = 0$ if $S(x) \leq \mu - \delta$ for some $\delta > 0$ then for all $F \in \mathcal{F}$ the set

$$F_\tau := \left\{\Phi_{\tau(x)}(x) \,:\, x \in F\right\}$$

is also a member of $\mathcal{F}$.

For convenience, given a subset $\mathcal{V} \subseteq \mathcal{M}$ and $a \in \mathbb{R}$, let

$$K_{a,\mathcal{V}} := \text{crit}\, S \cap S^{-1}(a) \cap \mathcal{V}$$

denote the set of critical points of S in $\mathcal{V}$ at the level a.

Our main tool will be the following *mountain pass theorem*, whose statement is similar to that of [5, Proposition 6.3]. In what follows, a *strict local minimizer* of a function $S : \mathcal{M} \to \mathbb{R}$ is a point $x \in \mathcal{M}$ such that there exists a neighborhood $\mathcal{N}$ of x such that $S(y) > S(x)$ for all $y \in \mathcal{N} \backslash \{x\}$.

5.1. THEOREM. *Let $\mathcal{M}$ be a Riemannian Hilbert manifold and $S : \mathcal{M} \to \mathbb{R}$ a function of class C^2. Suppose we are given a sequence $(\mathcal{F}_n)$ of families of subsets of $\mathcal{M}$ with $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for all $n \in \mathbb{N}$. Set $\mathcal{F}_\infty := \bigcup_n \mathcal{F}_n$. Set*

$$\mu_\infty := \inf_{F \in \mathcal{F}_\infty} \sup_{x \in F} S(x).$$

Suppose in addition that:

(1) *$\mathcal{F}_\infty$ is S-forward invariant, and the sets $F \in \mathcal{F}_\infty$ are connected;*
(2) *$\mu_\infty \in \mathbb{R}$;*
(3) *the flow Φ_s of $-\nabla S$ is relatively complete on $\{\mu_\infty - \eta \leq S \leq \mu_\infty + \eta\}$ for some $\eta > 0$;*
(4) *there are closed subsets $(\mathcal{U}_h)$ of $\mathcal{M}$ such that for all $\varepsilon > 0$, there exists $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ there exists $F \in \mathcal{F}_n$ and $0 < \varepsilon_1(n) < \varepsilon$ such that*

$$F \subseteq \{S \leq \mu_\infty - \varepsilon_1(n)\} \cup (\mathcal{U}_n \cap \{S \leq \mu_\infty + \varepsilon\})\,;$$

(5) *there are closed subsets* $(\mathcal{V}_n)$ *and a sequence* $(r_n) \subseteq \mathbb{R}^+$ *such that*

$$\mathcal{B}_{r_n}(\mathcal{U}_n) := \{x \in \mathcal{M} \,:\, \text{dist}(x, \mathcal{U}_n) < r_n\} \subseteq \mathcal{V}_n,$$

and such that $S|_{\mathcal{V}_n}$ *satisfies the Palais-Smale condition at the level* μ_∞.

Then if

$$\mathcal{V}_\infty := \bigcup_{n \in \mathbb{N}} \mathcal{V}_n,$$

S *has a critical point* $x \in \mathcal{V}_\infty$ *with* $S(x) = \mu_\infty$*, that is,*

$$K_{\mu_\infty, \mathcal{V}_\infty} \neq \emptyset.$$

Moreover if

$$\sup_{F \in \mathcal{F}_\infty} \inf_{x \in F} S(x) < \mu_\infty \tag{5.1}$$

then there is a point in $K_{\mu_\infty, \mathcal{V}_\infty}$ *which is not a strict local minimizer of* S.

The proof is an easy application of the following result, which can be found as [5, Lemma 6.2].

5.2. LEMMA. *Let* $\mathcal{M}$ *be a Riemannian Hilbert manifold and* $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{M}$ *closed subsets such that* $\mathcal{B}_r(\mathcal{U}) \subseteq \mathcal{V}$ *for some* $r > 0$*. Let* $S : \mathcal{M} \to \mathbb{R}$ *be a* C^2 *function, and let* $\mu \in \mathbb{R}$ *be such that* $S|_{\mathcal{V}}$ *satisfies the Palais-Smale condition at the level* μ*. Suppose in addition that the flow* Φ_s *of* $-\nabla S$ *is relatively complete on* $\{|S - \mu| \leq \eta\}$ *for some* $\eta > 0$*.*

Then if $\mathcal{N}$ *is any neighborhood of* $K_{\mu, \mathcal{V}}$ *relative to* $\mathcal{V}$*, for any* $\lambda > 0$ *there exists* $0 < \varepsilon < \delta < \lambda$ *such that for any* $0 < \varepsilon_1 < \varepsilon$ *there exists an admissible time* τ *such that*

$$\tau(x) = 0 \quad \textit{for all } x \in \{|S - \mu| \geq \delta\},$$

and such that if

$$F := \{S \leq \mu - \varepsilon_1\} \cup (\mathcal{U} \cap \{S \leq \mu + \varepsilon\}),$$

then

$$F_\tau \subseteq \mathcal{N} \cup \{S \leq \mu - \varepsilon_1\}.$$

Proof. (of Theorem 5.1)

We will show that $K_{\mu_\infty, \mathcal{V}_n} \neq \emptyset$ for n large enough. Fix $0 < \varepsilon < \delta < \lambda := 1$ as in the statement of Lemma 5.2. By hypothesis there exists $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ there exists $0 < \varepsilon_1(n) < \varepsilon$ and $F \in \mathcal{F}_n$ such that

$$F \subseteq \{S \leq \mu_\infty - \varepsilon_1(n)\} \cup (\mathcal{U}_n \cap \{S \leq \mu_\infty + \varepsilon\}).$$

For such n, $K_{\mu_\infty, \mathcal{V}_n} \neq \emptyset$. Indeed, if $K_{\mu_\infty, \mathcal{V}_n} = \emptyset$, by Lemma 5.2, there exists an admissible time τ such that $\tau \equiv 0$ on $\{S \leq \mu_\infty - \delta\}$, and such that F_τ satisfies

$$F_\tau \subseteq \{S \leq \mu_\infty - \varepsilon_1(n)\}$$

(for we may take $\mathcal{N} = \emptyset$ in the statement of Lemma 5.2). Since $\mathcal{F}_\infty$ is forward invariant, $F_\tau \in \mathcal{F}_\infty$. This contradicts the definition of μ_∞.

To prove the last statement, suppose that $K_{\mu_\infty,\mathcal{V}_\infty}$ consists entirely of strict local minimizers, and (5.1) holds. Choose $\lambda_0 > 0$ such that

$$\sup_{F\in\mathcal{F}_\infty} \inf_{x\in F} S(x) < \mu_\infty - 2\lambda_0.$$

For each $x \in K_{\mu_\infty,\mathcal{V}_\infty}$, let $\mathcal{N}(x)$ denote a neighborhood of x such that $S(y) > S(x)$ for all $y \in \mathcal{N}(x)\backslash\{x\}$, and let

$$\mathcal{N}_0 := \bigcup_{x\in K_{\mu_\infty,\mathcal{V}_\infty}} \mathcal{N}(x)$$

and $\mathcal{N}_n := \mathcal{N}_0 \cap \mathcal{V}_n$ for each $n \in \mathbb{N}$. Let $0 < \varepsilon < \delta < \lambda_0$ be given by Lemma 5.2 for $\mathcal{N}_0$. By hypothesis there exists $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ there exists $0 < \varepsilon_1(n) < \varepsilon$ and $F \in \mathcal{F}_n$ such that

$$F \subseteq \{S \leq \mu_\infty - \varepsilon_1(n)\} \cup (\mathcal{U}_n \cap \{S \leq \mu_\infty + \varepsilon\}).$$

By Lemma 5.2, there exists an admissible time τ such that $\tau \equiv 0$ on $\{S \leq \mu_\infty - \delta\}$ and such that

$$F_\tau \subseteq \mathcal{N}_n \cup \{S \leq \mu_\infty - \varepsilon_1(n)\} \subseteq \mathcal{N}_0 \cup \{S \leq \mu_\infty - \varepsilon_1(n)\}.$$

By definition of $\mathcal{N}_0$, the sets $\mathcal{N}_0$ and $\{S \leq \mu_\infty - \varepsilon_1(n)\}$ are disjoint, so $\mathcal{N}_0 \cup \{S \leq \mu_\infty - \varepsilon_1(n)\}$ is disconnected. Since F_τ is connected by hypothesis, we either have $F_\tau \subseteq \mathcal{N}_0$ and $F_\tau \cap \{S \leq \mu_\infty - \varepsilon_1(n)\} = \emptyset$, or $F_\tau \subseteq \{S \leq \mu_\infty - \varepsilon_1(n)\}$. The former fails since $\varepsilon_1(n) < \varepsilon < \lambda_0$, and the value of S decreases under Φ_s, and the latter contradicts the definition of μ_∞. The proof is complete. $\square$

Proof of the second statement of Theorem 1.1.

The main tool we will use in the proof of the second statement of Theorem 1.1 will be Theorem 5.1. The first step however is the following result, whose statement and proof closely parallel [5, Proposition C].

5.3. PROPOSITION. *Let $k \in \mathbb{R}^+$. Then there exists a constant $\mu_0 > 0$ such that if $f : I \to \Lambda_0 \times \mathbb{R}^+$ is any path such that, writing $f(0) = (x_0, T_0)$ and $f(1) = (x_1, T_1)$, it holds that:*

(1) $S_k(x_0, T_0) < 0$*;*
(2) x_1 *is the constant curve* $x_1(t) \equiv x_0(0)$*;*

then

$$\sup_{s\in I} S_k(f(s)) > \mu_0 > 0.$$

Remark. It is important to note that the constant μ_0 does *not* depend on T_1.

We shall need the following lemma, taken from [5, Lemma 5.1], in the proof of Proposition 5.3. As before, in the statement of the lemma, $l(x) := \int_0^1 |\dot{x}(t)|\, dt$.

5.4. LEMMA. *Let $\theta \in \Omega^1(\tilde{M})$. Given any $q \in \tilde{M}$ and any open neighborhood $V \subseteq \tilde{M}$ of q, there exists an open neighborhood $W \subseteq V$ of q and a constant $\beta > 0$ such that for any closed curve $x : [0,1] \to W$ it holds that*

$$\left| \int_x \theta \right| \leq \beta l(x)^2.$$

Proof. *(of Proposition 5.3)*

Fix a point $q \in M$, and choose a neighborhood $W \subseteq M$ of q small enough such that the conclusion of the lemma above holds. Pick $\rho \in \mathbb{R}^+$ such that

$$0 < \rho < \min\left\{\frac{1}{2}\text{diam}\, W, \sqrt{\frac{k}{2\beta^2}}\right\}.$$

Write $f(s) = (x_s, T_s)$, so $x_s \in \Lambda_0$ for all s. We claim that there exists $s_0 \in (0,1)$ such that $l(x_{s_0}) = \rho$. Since the functional $s \mapsto l(x_s)$ is continuous and $l(x_0) = 0$ it suffices to show that there exists $s_1 \in [0,1)$ such that $l(x_{s_1}) > \rho$.

If there exists $s_1 \in [0,1)$ such that $x_{s_1}(I) \subsetneq W$ then we are done, since then

$$l(x_{s_1}) \geq \text{dist}(q, W^c) > \frac{1}{2}\text{diam}\, W > \rho.$$

The other possibility is that $x_s(I) \subseteq W$ for all $s \in I$. In this case we claim that we may take $s_1 = 0$, that is, $l(x_0) > \rho$. By assumption if $y_0(t) = x_0(t/T_0)$ we have

$$\begin{aligned} 0 > S_k(x_0, T_0) &= \int_0^1 \frac{1}{2T_0} \left|\dot{x}_0(t)\right|^2 dt + kT_0 - \int_{C(x_0)} \sigma \\ &\geq \frac{1}{2T_0} l(x_0)^2 + kT_0 - \left|\int_{x_0} \theta\right| \\ &\geq \left(\frac{1}{2T_0} - \beta\right) l(x_0)^2 + kT_0, \end{aligned} \tag{5.2}$$

where the second inequality came from (3.1) and the third from the previous lemma. From this it follows that $T_0 > \frac{1}{2\beta}$, and thus

$$l(x_0)^2 > \frac{kT_0}{\beta - \frac{1}{2T_0}} > \frac{k}{2\beta^2} > \rho^2.$$

and we are done as before.

We now claim that $S_k(f(s_0)) > 0$, which will complete the proof. Since $x_{s_0} \in C_M^{\text{ac}}(q,q)$ and $l(x_{s_0}) < \frac{1}{2}\text{diam}\, W$, we have $x_{s_0}(I) \subseteq W$. In particular, (5.2) holds for x_{s_0} and so we have

$$S_k(f(s_0)) \geq \left(\frac{1}{2T_{s_0}} - \beta\right)\ell^2 + kT_{s_0} = P(T_{s_0}) \geq \min_{t \in \mathbb{R}^+} P(t),$$

where

$$P(t) := \left(\frac{1}{2t} - \beta\right)\rho^2 + kt.$$

It is elementary to see that

$$\min_{t \in \mathbb{R}^+} P(t) = \sqrt{\frac{\rho^2}{2k}} =: \mu_0 > 0,$$

and this completes the proof. □

The next lemma will be needed in order to prove relative completeness of the flow of $-\nabla S_k$ on any interval not containing zero.

5.5. LEMMA. *There exists a constant* $C > 0$ *such that for any* $(x_0, T_0) \in \Lambda_M \times \mathbb{R}^+$ *and any* $r > 0$*, if* $(x_1, T_1) \in \Lambda_M \times \mathbb{R}^+$ *satisfies*

$$\mathrm{dist}((x_0, T_0), (x_1, T_1)) < r,$$

then

$$|T_0 - T_1| < r$$

and

$$\mathrm{dist}_{\mathrm{HD}}(x_0, x_1) < Cr.$$

This result is essentially proved in [5, Lemma 2.3]; there a different metric is used on $\Lambda_M \times \mathbb{R}^+$ which means an additional condition must be imposed in the statement of the lemma. In our situation, since we are working with the standard metric (2.1) on $\Lambda_M \times \mathbb{R}^+$ this additional condition is not needed, and the proof in [5] goes through without any changes.

5.6. COROLLARY. *Let* $K \subseteq \tilde{M}$ *and* $B > 0$*. Let*

$$\mathcal{U} := \left\{(x, T) \in \Lambda_0^K \times \mathbb{R}^+ \,:\, T \le B\right\}.$$

Let C *be as in the statement of Lemma 5.5. Then if* $L \subseteq \tilde{M}$ *satisfies*

$$\left\{q \in \tilde{M} \,:\, \mathrm{dist}_{\tilde{g}}(q, q') \le Cr \text{ for some } q' \in K\right\} \subseteq L$$

and we set

$$\mathcal{V} := \left\{(x, T) \in \Lambda_0^L \times \mathbb{R}^+ \,:\, T \le B + r\right\}$$

then

$$\mathcal{B}_r(\mathcal{U}) \subseteq \mathcal{V}.$$

Proof. Suppose $(x_1, T_1) \in \mathcal{B}_r(\mathcal{U})$. Then there exists $(x_0, T_0) \in \mathcal{U}$ with

$$\mathrm{dist}((x_0, T_0), (x_1, T_1)) < r.$$

By the previous lemma,

$$\mathrm{dist}_{\mathrm{HD}}(x_0, x_1) < Cr$$

and $|T_0 - T_1| < r$. Thus $(x_1, T_1) \in \mathcal{V}$. □

Next, we prove relative completeness of the flow of $-\nabla S_k$ on any interval that doesn't contain zero. This proof is very similar to [5, Lemma 6.9].

5.7. LEMMA. *For all* $k \in \mathbb{R}^+$*, if* $[a, b] \subseteq \mathbb{R}$ *is an interval such that* $0 \notin [a, b]$ *then the local flow of* $-\nabla S_k$ *is relatively complete on* $(\Lambda_0 \times \mathbb{R}^+) \cap \{a \le S_k \le b\}$.

Proof. Let $\Phi_s : \Lambda_M \times \mathbb{R}^+ \to \Lambda_M \times \mathbb{R}^+$ denote the local flow of the vector field $-\nabla S_k$. Then for any $(x, T) \in \Lambda_M \times \mathbb{R}^+$,

$$S_k(\Phi_{s_1}(x, T)) - S_k(\Phi_{s_2}(x, T)) = \int_{s_1}^{s_2} |\nabla S_k(\Phi_s(x, T))|^2 \, ds.$$

By Cauchy-Schwarz inequality we see that

$$\begin{aligned}\operatorname{dist}(\Phi_{s_1}(x,T),\Phi_{s_2}(x,T))^2 &\leq \left(\int_{s_1}^{s_2}|\nabla S_k(\Phi_s(x,T))|\,ds\right)^2\\ &\leq |s_1-s_2|\int_{s_1}^{s_2}|\nabla S_k(\Phi_s(x,T))|^2\,ds,\end{aligned}$$

and hence

$$\operatorname{dist}(\Phi_{s_1}(x,T),\Phi_{s_2}(x,T))^2 \leq |s_1-s_2|\,|S_k(\Phi_{s_1}(x,T))-S_k(\Phi_{s_2}(x,T))|\,. \tag{5.3}$$

Now suppose we are given a pair $(x,T)\in\Lambda_0\times\mathbb{R}^+$, such that there exists $a,b\in\mathbb{R}$ with $0\notin[a,b]$ and

$$a\leq S_k(\Phi_s(x,T))\leq b \quad \text{for all } s \text{ such that } \Phi_s(x,T) \text{ is defined.}$$

Let $[0,\alpha)$ be the maximum interval of definition of $s\mapsto\Phi_s(x,T)$, where $\alpha\in(0,\infty]$. To complete the proof we need to show $\alpha=\infty$. Suppose to the contrary.

Write $\Phi_s(x,T)=(x_s,T_s)$. If $s_n\uparrow\alpha$ then $(\Phi_{s_n}(x,T))=:(x_n,T_n)$ is a Cauchy sequence by (5.3) in $(\Lambda_0\times\mathbb{R}^+)\cap\{a\leq S_k\leq b\}$. Thus $T_\alpha:=\lim_{s\uparrow\alpha}T_s$ exists and is finite.

If $T_\alpha>0$ then since the sequence (x_n,T_n) is Cauchy,

$$(x_\alpha,T_\alpha):=\lim_{n\to\infty}(x_n,T_n)$$

exists and is equal to $\Phi_\alpha(x,T)$. Since S_k is C^2 we can extend the solution $s\mapsto\Phi_s(x,T)$ at $s=\alpha$, contradicting the definition of α. Thus $T_\alpha=0$. Hence there exists a sequence $s_m\uparrow\alpha$ such that

$$\frac{d}{ds}T_{s_m}\leq 0.$$

As before write $x_m:=x_{s_m}$ and $T_m:=T_{s_m}$. By (5.3) and Lemma 5.5 we may assume there exists a compact set $K\subseteq\tilde{M}$ such that $(x_m,T_m)\subseteq\Lambda_0^K\times\mathbb{R}^+$ for all m. If $y_m(t):=x_m(t/T_m)$ then

$$\begin{aligned}0&\geq\frac{d}{ds}T_m\\ &=-\frac{\partial}{\partial T}S_k(x_m,T_m)\\ &=\frac{1}{T_m}\int_0^{T_m}\{-k+2E(y_m,\dot{y}_m)\}dt\\ &=-k+\frac{2e_m}{T_m},\end{aligned}$$

where the penultimate equality used (2.6). Since $\lim_{m\to\infty}T_m=0$, this forces $\lim_{m\to\infty}e_m=0$. As in the proof of the second part of Theorem 3.2, this implies $S_k(x_m,T_m)\to 0$, contradicting the fact that $0\notin[a,b]$. This implies that we must have originally had $\alpha=\infty$, and so completes the proof. $\square$

We now move towards proving the second statement of Theorem 1.1. In fact, we will prove the following stronger result, which is based on [5, Proposition 7.1].

5.8. PROPOSITION. *Let $c = c(g,\sigma) \in \mathbb{R} \cup \{\infty\}$. For almost all $k \in (0,c)$ there exists a contractible closed orbit of ϕ_t with energy k. Moreover this orbit has positive S_k-action, and is not a strict local minimizer of S_k on $\Lambda_0 \times \mathbb{R}^+$. This holds for a specific $k \in (0,c)$ if S_k is known to satisfy the Palais-Smale condition on the level k.*

Proof. Fix $k_0 \in \mathbb{R}^+$. There exists $(x_0,T_0) \in \Lambda_0 \times \mathbb{R}^+$ such that $S_{k_0}(x_0,T_0) < 0$. Indeed, there exists a closed curve $\tilde{y} : [0,T_0] \to \tilde{M}$ such that $A_{k_0}(\tilde{y}) < 0$. Then the projection $y : [0,T_0] \to M$ of $\tilde{y}$ to M is a closed curve, and if $x_0(t) := y(tT_0)$ then $(x_0,T_0) \in \Lambda_0 \times \mathbb{R}^+$ and $S_{k_0}(x_0,T_0) = A_{k_0}(\tilde{y}) < 0$. There exists $\varepsilon > 0$ such that for all $k \in J := [k_0, k_0+\varepsilon]$ we have $S_k(x_0,T_0) < 0$.

Let x_1 denote the constant loop at $x_0(0)$. Given $k \in J$, let $\mu_0(k) > 0$ be the constant given by Proposition 5.3 such that any path $f \in C^0(I, \Lambda_0 \times \mathbb{R}^+)$ with $f(0) = (x_0,T_0)$ and $f(1) = (x_1,T)$ for some $T > 0$ satisfies

$$\sup_{s\in I} S_k(f(s)) > \mu_0(k).$$

Choose $T_1 > 0$ such that

$$T_1 < \inf_{k\in J} \frac{\mu_0(k)}{k}.$$

Then

$$\max\{S_k(x_0,T_0), S_k(x_1,T_1)\} = kT_1 < \mu_0(k) \quad \text{for all } k \in J.$$

Set

$$\Gamma := \left\{ f \in C^0(I, \Lambda_0 \times \mathbb{R}^+) \, : \, f(0) = (x_0,T_0), f(1) = (x_1,T_1) \right\}.$$

Let $(K_n) \subseteq \tilde{M}$ denote compact sets such that $K_n \subseteq K_{n+1}$ and $\bigcup_n K_n = \tilde{M}$. Let

$$\Gamma_n := \Gamma \cap C^0(I, \Lambda_0^{K_n} \times \mathbb{R}^+).$$

Define for $k \in J$,

$$\mu_n(k) := \inf_{f\in\Gamma_n} \sup_{s\in I} S_k(f(s));$$
$$\mu_\infty(k) := \inf_{f\in\Gamma} \sup_{s\in I} S_k(f(s)).$$

Then $\mu_n(k) \geq \mu_{n+1}(k) \geq \mu_\infty(k) \geq \mu_0(k)$ for all $n \in \mathbb{N}$ and $k \in J$, and the functions $\mu_n : J \to \mathbb{R}$ converge pointwise to μ_∞. Moreover both μ_n and μ_∞ are non-decreasing. Since μ_∞ is non-decreasing, by Lebesgue's theorem there exists a subset $J_0 \subseteq (k_0, k_0+\varepsilon)$ with $J \backslash J_0$ having measure zero such that $\mu_\infty|_{J_0}$ is locally Lipschitz. In other words, for all $j \in J_0$ there exist constants $M(j) > 0$ and $\delta(j) > 0$ such that for all $|\delta| < \delta(j)$ it holds that

$$|\mu_\infty(j+\delta) - \mu_\infty(j)| < M(j)\,|\delta|.$$

Fix $j \in J_0$ and a sequence $(j_m) \subseteq J_0$ with $j_m \downarrow j$. Let $f_{n,m} \in \Gamma_n$ be paths such that

$$\max_{s\in I} S_{j_m}(f_{n,m}(s)) \leq \mu_n(j_m) + (j_m - j).$$

Next, define

$$\mathcal{U}_n := \left\{ (x,T) \in \Lambda_0^{K_n} \times \mathbb{R}^+ \, : \, T \leq M(j) + 2 \right\}.$$

Choose another collection $(L_n) \subseteq \tilde{M}$ of compact sets such that $K_n \subseteq L_n$, and such that if

$$\mathcal{V}_n := \left\{ (x,T) \in \Lambda_0^{L_n} \times \mathbb{R}^+ \, : \, T \leq M(j) + 3 \right\},$$

then $\mathcal{B}_1(\mathcal{U}_n) \subseteq \mathcal{V}_n$. Such a collection (L_n) exists by Corollary 5.6. Since $\mu_\infty(j) \neq 0$, from Proposition 3.7 it follows that $S_j|_{\mathcal{V}_n}$ satisfies the Palais-Smale condition at the level $\mu_\infty(j)$ for all $n \in \mathbb{N}$.

Since $k \mapsto S_k(x,T)$ is increasing,

$$\max_{s\in I} S_j(f_{n,m}(s)) \leq \max_{s\in I} S_{j_m}(f_{n,m}(s)) \leq \mu_n(j_m) + (j_m - j). \tag{5.4}$$

If $s \in I$ is such that

$$S_j(f_{n,m}(s)) > \mu_\infty(j) - (j_m - j),$$

writing $f_{n,m}(s) =: (x_s^{n,m}, T_s^{n,m})$ we have:

$$\begin{aligned} T_s^{n,m} &= \frac{S_{j_m}(f_{n,m}(s)) - S_j(f_{n,m}(s))}{j_m - j} \\ &\leq \frac{\mu_\infty(j_m) - \mu_n(j)}{j_m - j} + 2 \\ &\leq \frac{\mu_\infty(j_m) - \mu_\infty(j)}{j_m - j} + 2 \\ &\leq M(j) + 2, \end{aligned}$$

for n large enough.

Given $\varepsilon > 0$, first choose m large enough such that

$$j_m - j < \frac{\varepsilon}{2(M(j)+1)},$$

and then select n large enough so that

$$\mu_n(j_m) - \mu_\infty(j_m) < \varepsilon/2.$$

Then

$$\begin{aligned} \mu_n(j_m) - \mu_\infty(j) + (j_m - j) &= \{\mu_n(j_m) - \mu_\infty(j_m)\} + \{\mu_\infty(j_m) - \mu_\infty(j)\} + (j_m - j) \\ &< \varepsilon/2 + M(j)(j_m - j) + (j_m - j) \\ &< \varepsilon. \end{aligned}$$

Then by (5.4),

$$f_{n,m}(I) \subseteq \{S_j \leq \mu_\infty(j) - (j_m - j)\} \cap (\mathcal{U}_n \cap \{S_j \leq \mu_\infty(j) + \varepsilon\}).$$

Since $\mu_\infty(j) \neq 0$, by Lemma 5.7 the gradient flow of $-S_j$ is relatively complete on $\{\mu_\infty(j) - \eta \leq S_j \leq \mu_\infty(j) + \eta\}$ for some $\eta > 0$. Theorem 5.1 then gives a critical point for $S_j|_{\Lambda_0 \times \mathbb{R}^+}$ which is not a strict local minimizer (we are applying Theorem 5.1 with $\mathcal{F}_n := \{f(I) \,:\, f \in \Gamma_n\}$).

Finally, suppose that $k < c(g,\sigma) \leq \infty$ is such that S_k satisfies the Palais-Smale condition. In this case the theorem is immediate from Lemma 5.7 and Theorem 5.1. Indeed, by Lemma 5.7 we may simply take $\mathcal{U}_n = \mathcal{V}_n = \mathcal{V}_\infty = \Lambda_0 \times \mathbb{R}^+$, as then the hypotheses of Theorem 5.1 are trivially satisfied. This completes the proof of Theorem 1.1. □

REFERENCES

1. A. Abbondandolo and M. Schwarz, *A smooth pseudo-gradient for the Lagrangian action functional*, arXiv:0812.4364 (2008).
2. ______, *Estimates and computations in Rabinowitz-Floer homology*, arXiv:0907:1976 (2009).
3. K. Burns and G. P. Paternain, *Anosov magnetic flows, critical values and topological entropy*, Nonlinearity **15** (2002), 281–314.
4. K. Cieliebak, U. Frauenfelder, and A. Oancea, *Rabinowitz Floer homology and symplectic homology*, arXiv:0903.0768 (2009).
5. G. Contreras, *The Palais-Smale condition on contact type energy levels for convex Lagrangian systems*, Calculus of Variations **27** (2006), no. 3, 321–395.
6. G. Contreras, J. Delgado, and R. Iturriaga, *Lagrangian flows: The dynamics of globally minimizing orbits - II*, Bol. Soc. Bras. Mat. **28** (1997), no. 2, 155–196.
7. G. Contreras and R. Iturriaga, *Global Minimizers of Autonomous Lagrangians*, Colloqio Brasileiro de Matematica, vol. 22, IMPA, Rio de Janeiro, 1999.
8. G. Contreras, R. Iturriaga, G. P. Paternain, and M. Paternain, *The Palais-Smale Condition and Mañé's Critical Values*, Ann. Henri Poincaré **1** (2000), 655–684.
9. G. Contreras, L. Macarini, and G. P. Paternain, *Periodic Orbits for Exact Magnetic Flows on Surfaces*, Int. Math. Res. Not. **8** (2004), 361–387.
10. V. Ginzburg, *On closed trajectories of a charge in a magnetic field. An application of symplectic geometry*, Contact and symplectic geometry (Cambridge, 1994) (C. B. Thomas, ed.), Publications of the Newton Institute, vol. 8, Cambridge University Press, 1996, pp. 131–148.
11. V. Ginzburg and B. Z. Gürel, *Periodic orbits of twisted geodesic flows and the Weinstein-Moser Theorem*, Comment. Math. Helv. **84** (2009), 865–907.
12. E. Kerman, *Symplectic geometry and the motion of a particle in a magnetic field*, Ph.D. thesis, University of California at Santa Cruz, 2000.
13. G. Lu, *Finiteness of the Hofer-Zehdner capacity of neighborhoods of symplectic submanifolds*, Int. Math. Res. Not. (2006), 1–33.
14. R. Mañé, *Lagrangian flows: the dynamics of globally minimizing orbits*, International Congress on Dynamical Systems in Montevideo (a tribute to Ricardo Mañé) (J. Lewowicz F. Ledrappier and S. Newhouse, eds.), Pitman Research Notes in Math., vol. 362, Longman, 1996, pp. 120–131.
15. L. Macarini, *Hofer-Zehnder capacity and Hamiltonian circle actions*, Commum. Contemp. Math. **6** (2006), 913–945.
16. W. J. Merry, *On the Rabinowitz Floer homology of twisted cotangent bundles*, In preparation (2009).
17. O. Osuna, *Periodic orbits of weakly exact magnetic flows*, Preprint (2005).
18. G. P. Paternain, *Magnetic Rigidity of Horocycle Flows*, Pacific J. Math. **225** (2006), 301–323.
19. L. Polterovich, *Geometry on the group of Hamiltonian diffeomorphisms*, Doc. Math. **2** (1998), 401–410.
20. F. Schlenk, *Applications of Hofer's geometry to Hamiltonian dynamics*, Comment. Math. Helv. **81** (2006), 105–121.
21. M. Usher, *Floer homology in disc bundles and symplectically twisted geodesic flows*, J. Modern Dynamics **3** (2009), no. 1, 61–101.

DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS, UNIVERSITY OF CAMBRIDGE, CAMBRIDGE CB3 0WB, ENGLAND

E-mail address: `w.merry@dpmms.cam.ac.uk`